OSCILLATION PROPERTIES OF CERTAIN TYPES OF FIRST ORDER NEUTRAL DELAY DIFFERENCE EQUATIONS

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ABSTRACT

In this paper some sufficient condition for the oscillation of first order neutral delay difference equation were obtained.

KEYWORDS :

Neutral Delay Difference Equation, Oscillation, Nonoscillation, Eventually positive.

Introduction 1.1

In this paper some sufficient condition for the oscillation of first order neutral delay difference equation of the form

$$\Delta\,(a_n x_n - p_n x_{n-k}) + q_n\, f\,(x_{n-l}) = 0, n \in N(n_0) \qquad (1.1.1)$$

and

$$\Delta\,(x_n + p_n x_{n-k}) + q_n\, f\,(x_{n-l}) = 0, n \in N(n_0) \qquad (1.1.2)$$

were obtained with the assumption of e the following conditions.

H$_1$: $\{p_n\}$ is an positive sequence.

H$_2$: f is a continuous function such that $uf(u) \geq 0$.

H$_3$: If there exists a function w such that w(u)>0, for u>0 and $f(uv) \leq w(u)\left|f(v)\right|$.

H$_4$: If there exists a function ϕ such that $\phi(u)$ is increasing and $u\phi(u) > 0$, for $u \neq 0$ & $\left|\phi(u+v)\right| \leq \left|f(u)\, f(v)\right|$.

1.2 Existence of Oscillatory Solutions

In this section, I obtain some sufficient condition for the oscillatory solutions of the equation (1.1.1) and (1.1.2)

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Theorem 1. 2.1

Assume that $\frac{p_n}{a_{n-k}} \le 1$ and x_n be an eventually positive solution of the equation (1.1.1) and

$y_n = (a_n x_n - p_n x_{n-k})$. Then eventually $y_n > 0$.

Proof

Let us consider $x_n > 0$, $x_{n-l} > 0$, $x_{n-k} > 0$ for some $n > n_1$.

From the equation (1.1.1),

$\Delta y_n = -q_n f(x_{n-l}) < 0.$ Hence y_n is a decreasing function.

Suppose y_n is not eventually positive, then eventually $y_n < 0$.

Hence there exists $n_2 > n_1$ and $M>0$, such that $y_n < -M$.

Let, $z_n = a_n x_n > 0.$

Then, $z_n = y_n + p_n x_{n-k}.$

$$z_n < -M + \frac{p_n}{a_{n-k}} z_{n-k} .$$

Hence $z_n \to -\infty$, $n \to \infty$ as $n \to \infty$. Which contradicts the fact that z_n is eventually positive. Hence the proof.

Theorem 1.2.2

Assume that $p_n, q_n > 0$ and $\frac{p_n}{a_{n-k}} \le 1.$

If, $\lim_{n\to\infty} \inf q_n^* \left(1 + \frac{p_{n-l}\lambda_{n-k}}{q_{n-k}}\right) > 0,$

where $q_n^* = \frac{q_n}{a_{n-l}}$,then every solution of equation (1.1.1) is an oscillatory solution.

Proof

Let us assume the contradiction that equation (1.1.1) has an non oscillatory solution. Let us consider x_n is eventually positive.

Let us consider $x_n > 0$, $x_{n-l} > 0$, $x_{n-k} > 0$ for some $n > n_1$.

By theorem 1.2.1, y_n is eventually positive.

Also we have

$$\Delta y_n = -q_n f(x_{n-l})$$

$$y_n = a_n x_n - p_n x_{n-k}$$

$$\Delta y_n \le -q_n x_{n-l} \qquad (1.2.1)$$

$$\Delta y_n \le -q_n \frac{y_{n-l} + p_{n-l} x_{n-k-l}}{a_{n-l}}$$

$$\Delta y_n = \frac{-q_n y_{n-l}}{a_{n-l}} - \frac{q_n p_{n-l} x_{n-k-l}}{a_{n-l}}$$

From the equation (1.2.1),

$$\Delta y_n \le \frac{-q_n y_{n-l}}{a_{n-l}} + \frac{q_n p_{n-l} \Delta y_{n-k}}{a_{n-l} q_{n-k}}$$

Hence y_n satisfies the inequality,

$$\Delta y_n + \frac{q_n y_{n-l}}{a_{n-l}} - \frac{q_n p_{n-l} \Delta y_{n-k}}{a_{n-l} q_{n-k}} \le 0.$$

Let $\lambda_n = \dfrac{-\Delta y_n}{y_n}$, then

$$\lambda_n y_n \ge \frac{q_n y_{n-l}}{a_{n-l}} - \frac{q_n p_{n-l} \Delta y_{n-k}}{a_{n-l} q_{n-k}}$$

$q_n{}^* = \dfrac{q_n}{a_{n-l}}$, Hence we have,

$$\lambda_n \ge q_n^* + \frac{p_{n-l} \lambda_{n-k} q_n^*}{q_{n-k}}$$

Hence $\lim_{n\to\infty} \inf q_n^* (1 + \frac{p_{n-l}\lambda_{n-k}}{q_{n-k}}) \le \lambda_n,$

Therefore, $\lim_{n\to\infty} \inf q_n^* (1 + \frac{p_{n-l}\lambda_{n-k}}{q_{n-k}}) \le 0.$

Which contradicts the given condition of the theorem. Hence every solution of the equation (1.1.1) is an oscillatory solution.

Theorem 1.2.3

Suppose that $a_n = 1, for n = 1,2,3....$ Then $\Delta\,(x_n + p_n x_{n-k}) + q_n f(x_{n-l}) = 0, n \in N(n_0)$ is oscillatory if there exists a function λ such that $0 \le \lambda_n \le 1$ for $n \ge n_0$ and the differenc inequality $\Delta\, z_n + Q_n \phi(z_{n-l+k}) \le 0,$ (1.2.2)

has oscillatory solution where,

$$Q_n = \min\left(\lambda_n q_n, \frac{(1-\lambda_{n-k}) q_{n-l}}{w p_{n-l}}\right)$$

Proof

Suppose to the contrary that there is a non oscillatory solution x_n. Assume that ,

$$x_n > 0,$$

For all $n>n_0$. Let

$$y_n = x_n + p_n x_{n-k}$$

$$\Delta(y_n) = -q_n f(x_{n-l}) < 0$$

Also $y_{n+1}<y_n$, y_n is decreasing function.,

Hence $y_{n+1} + q_n f(x_{n-l}) = y_n$

$$y_n > q_n f(x_{n-l}), n \geq n_0.$$

Taking summation from n_0 to m , $m>n_0$,

$$\sum_{n=n_0}^{m} y_n > \sum_{n=n_0}^{m} q_n f(x_{n-l})$$

$$\sum_{n=n_0}^{m} y_n > \sum_{n=n_0}^{m}((\lambda_n q_n f(x_{n-l}) + (1-\lambda_n)q_n f(x_{n-l}))$$

$$\sum_{n=n_0}^{m} y_n > \sum_{n=n_0}^{m} Q_n f(x_{n-l}) + \sum_{n=n_0+k}^{m}(1-\lambda_{n-k})q_{n-k} f(x_{n-k-l})$$

$$\sum_{n=n_0}^{m} y_n > \sum_{n=n_0}^{m} Q_n f(x_{n-l}) + \sum_{n=n_0+k}^{m} Q_n w(p_{n-l}) f(x_{n-k-l})$$

$$\sum_{n=n_0}^{m} y_n > \sum_{n=n_0}^{m} Q_n f(x_{n-l}) + \sum_{n=n_0+k}^{m} Q_n f(p_{n-l} x_{n-k-l})$$

$$\sum_{n=n_0}^{m} y_n > \sum_{n=n_0}^{m} Q_n \{ f(x_{n-l}) + f(p_{n-l} x_{n-k-l})\}$$

$$\sum_{n=n_0}^{m} y_n > \sum_{n=n_0+k}^{m} Q_n \{\phi(x_{n-l} + p_{n-l} x_{n-k-l})\}$$

$$\sum_{n=n_0}^{m} y_n > \sum_{n=n_0+k}^{m} Q_n \{y_{n-l}\}$$

Let $z_n = \sum_{n_0+k}^{m} Q_n \phi\{y_{n-l}\} > 0.$

Then $\Delta z_n = z_{n+1} - z_n$

$$\Delta z_n = \sum_{n=n_0+k}^{m}(Q_{n+1}\phi\{y_{n+1-l}\} - Q_n \phi(y_{n-l}))$$

$$\Delta z_n = Q_{m+1}\phi(y_{m+1-l}) - \phi_{n_0+k}\phi(y_{n_0+k-l})$$

$$\Delta z_n > -Q_n \phi(y_{n-l})$$

$$\Delta z_n > -Q_n \phi(z_{n-l+k})$$

$\Delta z_n + Q_n \phi(z_{n-l+k}) > 0.$ This condition holds when z_n is eventually positive solution. This is a contradiction to the equation (1.2.2).

Hence the proof complte s. Similarly we prove that, when x_n is eventually negative.

2.1 Examples

Example 2.1.1

Consider the first order neutral delay difference equation

$$\Delta\ (nx_n - x_{n-1}) + (2n+3)x_{n-2}{}^3 = 0, n > 0$$

Here

$$a_n = n, k = 1, l = 2, p_n = -1, q_n = (2n+3)$$

All the conditions of the theorem 1.2.2 are satisfied.

Hence all its solutions are oscillatory. One such solution is $(-1)^n$.

Example 2.1.2

Consider the first order neutral delay difference equation

$$\Delta\ (x_n - \frac{1}{n-1}x_{n-1}) + \frac{2n+3}{(n-2)^3}x_{n-2}{}^3 = 0, n > 2$$

Here

$$a_n = 1, k = 1, l = 2, p_n = -\frac{1}{n-1}, q_n = \frac{2n+3}{(n-2)^3}$$

Hence all the conditions of the theorem 1.2.2 are satisfied.

Hence all its solutions are oscillatory. One such solution is $n(-1)^n$.

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